DARBOUX'S FORMULA WITH INTEGRAL REMAINDER OF FUNCTIONS WITH TWO INDEPENDENT VARIABLES

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ABSTRACT. In the article, the noted Darboux's formula of functions with single variable is generalized to that of functions of two independent variables with integral remainder, some important special cases of Darboux's formula of functions with two variables are obtained, and some estimates of the integral remainders and Darboux's expansion of the function $\ln(x+y)$ are given. These results generalize A. Sard's formula in numerical integration.

1. INTRODUCTION

The well-known G. Darboux's formula of functions with single variable was given in [13, p. 217], which can be modified slightly as follows

Theorem A. *Suppose $f(x)$ is defined on an interval $I \subset \mathbb{R}$ and $f^{(n)}(x)$ is absolutely continuous on I. Let $P_n(t)$ be a polynomial of degree n and the coefficient of the term t^n equal a_n and $a \in I$. Then*

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{n!a_n}\left[P_n^{(n-k)}(x)f^{(k)}(x) - P_n^{(n-k)}(a)f^{(k)}(a)\right] + \frac{(-1)^n}{n!a_n}\int_a^x P_n(t)f^{(n+1)}(t)\,\mathrm{d}t. \quad (1.1)$$

Remark 1. If letting $P_n(t) = (t-x)^n$ in (1.1), we can obtain the following Taylor's formula with integral remainder

$$f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i + \int_a^x (x-t)^n f^{(n+1)}(t)\,\mathrm{d}t. \quad (1.2)$$

Replacing $P_n(t)$ by $P_{m+n}(t) = (t-x)^m(t-a)^n$ in Theorem A, we have the following

Theorem B (Obreschkoff's formula of functions with single variable [13, p. 218])**.** *Suppose that $f(x)$ is defined on an interval $I \subset \mathbb{R}$ and $f^{(m+n)}(x)$ is absolutely continuous on I. Let $a \in I$, then*

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k \frac{\binom{n}{k}}{\binom{m+n}{k}}\frac{(x-a)^k}{k!}f^{(k)}(x) = \sum_{k=0}^{m}\frac{\binom{m}{k}}{\binom{m+n}{k}}\frac{(x-a)^k}{k!}f^{(k)}(a)&\\ + \frac{1}{(m+n)!}\int_a^x (x-t)^m(a-t)^n f^{(m+n+1)}(t)\,\mathrm{d}t.& \quad (1.3)\end{aligned}$$

Remark 2. Theorem B can be applied to obtain H. Padé approximations of the functions e^x, x^α, $\ln x$, and $\arctan x$. See [12, p. 191–205].

In this paper, we will adopt the following notations used in [11]:

$$\begin{aligned} f^{(i,j)}(x,y) &= \frac{\partial^{i+j}f(x,y)}{\partial x^i \partial y^j},\\ f^{(0,0)}(x,y) &= f(x,y),\\ f^{(i,j)}(\alpha,\beta) &= f^{(i,j)}(x,y)\big|_{(x,y)=(\alpha,\beta)}. \end{aligned} \quad (1.4)$$

where $i,j \geq 0$ are integers and $(\alpha,\beta) \in D \subset \mathbb{R}^2$.

A. H. Stroud has pointed out in his celebrated book [11, p. 138] that one of the most important tools in the numerical integration of double integrals is the following Taylor-like expansion due to A. Sard [10]:

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Theorem C (Sard's formula of functions with two variables [11, p. 138 and p. 157])**.** *Suppose that $f(x,y)$ is defined on a convex region $D \subset \mathbb{R}^2$, $(a,b) \in D$, and $f^{(i,j)}(x,y)$ is continuous on D for $i+j \le m$. Then*

$$\begin{aligned} f(x,y) = {} & \sum_{i+j<m} \frac{(x-a)^i}{i!} \cdot \frac{(y-b)^j}{j!} f^{(i,j)}(a,b) \\ & + \sum_{j<q} \frac{(y-b)^j}{j!} \int_a^x \frac{(x-u)^{m-j-1}}{(m-j-1)!} f^{(m-j,j)}(u,b)\,\mathrm{d}u \\ & + \sum_{i<p} \frac{(x-a)^i}{i!} \int_b^y \frac{(y-v)^{m-i-1}}{(m-i-1)!} f^{(i,m-i)}(a,v)\,\mathrm{d}v \\ & + \int_a^x \int_b^y \frac{(x-u)^{p-1}}{(p-1)!} \cdot \frac{(y-v)^{q-1}}{(q-1)!} f^{(p,q)}(u,v)\,\mathrm{d}u\,\mathrm{d}v, \end{aligned} \tag{1.5}$$

where i,j are nonnegative integers, p,q are positive integers, and $m=p+q$.

In [3], some new Taylor-like expansions of functions with two independent variables and estimates of their remainders are established.

In this paper, we generalize G. Darboux's formula of functions with single variable in Theorem A to that of functions with two independent variables as follows.

Theorem 1 (Darboux's formula of functions with two variables)**.** *Suppose $D \subset \mathbb{R}^2$ is a convex region and $(a,b) \in D$, define $f : D \to \mathbb{R}$ such that $f^{(i,j)}(x,y)$ is continuous on D for $0 \le i \le n$ and $0 \le j \le m$. Let $P_n(t)$ be a polynomial of degree n with coefficient a_n of the term t^n and $Q_m(s)$ a polynomial of degree m with coefficient b_m of the term s^m. Then*

$$f(x,y) = f(a,b) + C(f,P_n,Q_m) + D(f,P_n,Q_m) + S(f,P_n,Q_m) + T(f,P_n,Q_m), \tag{1.6}$$

where

$$\begin{aligned} C(f,P_n,Q_m) = {} & \sum_{k=1}^{n} \frac{(-1)^{k+1}}{n!a_n} \big[P_n^{(n-k)}(x) f^{(k,0)}(x,b) - P_n^{(n-k)}(a) f^{(k,0)}(a,b)\big] \\ & + \sum_{i=1}^{m} \frac{(-1)^{i+1}}{m!b_m} \big[Q_m^{(m-i)}(y) f^{(0,i)}(a,y) - Q_m^{(m-i)}(b) f^{(0,i)}(a,b)\big], \end{aligned} \tag{1.7}$$

$$\begin{aligned} D(f,P_n,Q_m) = {} & \sum_{k=1}^{n}\sum_{i=1}^{m} \frac{(-1)^{k+i}}{m!n!a_nb_m} P_n^{(n-k)}(x) \big[Q_m^{(m-i)}(y) f^{(k,i)}(x,y) - Q_m^{(m-i)}(b) f^{(k,i)}(x,b)\big] \\ & - \sum_{k=1}^{n}\sum_{i=1}^{m} \frac{(-1)^{k+i}}{m!n!a_nb_m} P_n^{(n-k)}(a) \big[Q_m^{(m-i)}(y) f^{(k,i)}(a,y) - Q_m^{(m-i)}(b) f^{(k,i)}(a,b)\big], \end{aligned} \tag{1.8}$$

$$\begin{aligned} S(f,P_n,Q_m) = {} & \frac{(-1)^n}{n!a_n} \int_a^x P_n(t) f^{(n+1,0)}(t,b)\,\mathrm{d}t + \frac{(-1)^m}{m!b_m} \int_b^y Q_m(s) f^{(0,m+1)}(a,s)\,\mathrm{d}s \\ & + \sum_{k=1}^{n} \frac{(-1)^{m+k+1}}{m!n!a_nb_m} \int_b^y Q_m(s) \big[P_n^{(n-k)}(x) f^{(k,m+1)}(x,s) - P_n^{(n-k)}(a) f^{(k,m+1)}(a,s)\big]\,\mathrm{d}t \\ & + \sum_{i=1}^{m} \frac{(-1)^{n+i+1}}{m!n!a_nb_m} \int_a^x P_n(t) \big[Q_m^{(m-i)}(y) f^{(n+1,i)}(t,y) - Q_m^{(m-i)}(b) f^{(n+1,i)}(t,b)\big]\,\mathrm{d}t, \end{aligned} \tag{1.9}$$

$$T(f,P_n,Q_m) = \frac{(-1)^{m+n}}{m!n!a_nb_m} \int_a^x \int_b^y P_n(t) Q_m(s) f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s. \tag{1.10}$$

Further, we discuss some important special cases of G. Darboux's formula (1.6) for functions with two independent variables and give some estimates of the integral remainders. As an application, we calculate G. Darboux's formula of the function $\ln(x+y)$ finally.

2. Proof of Theorem 1

Applying Theorem A to the function $f(x,y)$ with respect to the variable x yields

$$\begin{aligned} f(x,y) = f(a,y) & + \sum_{k=1}^{n} \frac{(-1)^{k+1}}{n!a_n} \big[P_n^{(n-k)}(x) f^{(k,0)}(x,y) - P_n^{(n-k)}(a) f^{(k,0)}(a,y)\big] \\ & + \frac{(-1)^n}{n!a_n} \int_a^x P_n(t) f^{(n+1,0)}(t,y)\,\mathrm{d}t. \end{aligned} \tag{2.1}$$

Similarly, applying Theorem A to the functions $f(a,y)$, $f^{(k,0)}(x,y)$, $f^{(k,0)}(a,y)$ and $f^{(n+1,0)}(t,y)$ with respect to the variable y gives us

$$\begin{aligned} f(a,y) = f(a,b) &+ \sum_{i=1}^{m} \frac{(-1)^{i+1}}{m!b_m} \bigl[Q_m^{(m-i)}(y) f^{(0,i)}(a,y) - Q_m^{(m-i)}(b) f^{(0,i)}(a,b)\bigr] \\ &+ \frac{(-1)^m}{m!b_m} \int_b^y Q_m(s) f^{(0,m+1)}(a,s)\,\mathrm{d}s, \end{aligned} \tag{2.2}$$

$$\begin{aligned} f^{(k,0)}(x,y) = f^{(k,0)}(x,b) &+ \sum_{i=1}^{m} \frac{(-1)^{i+1}}{m!b_m} \bigl[Q_m^{(m-i)}(y) f^{(k,i)}(x,y) - Q_m^{(m-i)}(b) f^{(k,i)}(x,b)\bigr] \\ &+ \frac{(-1)^m}{m!b_m} \int_b^y Q_m(s) f^{(k,m+1)}(x,s)\,\mathrm{d}s, \end{aligned} \tag{2.3}$$

$$\begin{aligned} f^{(k,0)}(a,y) = f^{(k,0)}(a,b) &+ \sum_{i=1}^{m} \frac{(-1)^{i+1}}{m!b_m} \bigl[Q_m^{(m-i)}(y) f^{(k,i)}(a,y) - Q_m^{(m-i)}(b) f^{(k,i)}(a,b)\bigr] \\ &+ \frac{(-1)^m}{m!b_m} \int_b^y Q_m(s) f^{(k,m+1)}(a,s)\,\mathrm{d}s, \end{aligned} \tag{2.4}$$

$$\begin{aligned} f^{(n+1,0)}(t,y) = f^{(n+1,0)}(t,b) &+ \sum_{i=1}^{m} \frac{(-1)^{i+1}}{m!b_m} \bigl[Q_m^{(m-i)}(y) f^{(n+1,i)}(t,y) - Q_m^{(m-i)}(b) f^{(n+1,i)}(t,b)\bigr] \\ &+ \frac{(-1)^m}{m!b_m} \int_b^y Q_m(s) f^{(n+1,m+1)}(t,s)\,\mathrm{d}s. \end{aligned} \tag{2.5}$$

Substituting formulas (2.2), (2.3), (2.4), and (2.5) into (2.1) and rearranging leads to

$$\begin{aligned} f(x,y) = f(a,b) &+ \sum_{k=1}^{n} \frac{(-1)^{k+1}}{n!a_n} \bigl[P_n^{(n-k)}(x) f^{(k,0)}(x,b) - P_n^{(n-k)}(a) f^{(k,0)}(a,b)\bigr] \\ &+ \sum_{i=1}^{m} \frac{(-1)^{i+1}}{m!b_m} \bigl[Q_m^{(m-i)}(y) f^{(0,i)}(a,y) - Q_m^{(m-i)}(b) f^{(0,i)}(a,b)\bigr] \\ &+ \sum_{k=1}^{n}\sum_{i=1}^{m} \frac{(-1)^{k+i}}{m!n!a_nb_m} P_n^{(n-k)}(x) \bigl[Q_m^{(m-i)}(y) f^{(k,i)}(x,y) - Q_m^{(m-i)}(b) f^{(k,i)}(x,b)\bigr] \\ &- \sum_{k=1}^{n}\sum_{i=1}^{m} \frac{(-1)^{k+i}}{m!n!a_nb_m} P_n^{(n-k)}(a) \bigl[Q_m^{(m-i)}(y) f^{(k,i)}(a,y) - Q_m^{(m-i)}(b) f^{(k,i)}(a,b)\bigr] \\ &+ \frac{(-1)^n}{n!a_n} \int_a^x P_n(t) f^{(n+1,0)}(t,b)\,\mathrm{d}t + \frac{(-1)^m}{m!b_m} \int_b^y Q_m(s) f^{(0,m+1)}(a,s)\,\mathrm{d}s \\ &+ \sum_{k=1}^{n} \frac{(-1)^{m+k+1}}{m!n!a_nb_m} \int_b^y Q_m(s) \bigl[P_n^{(n-k)}(x) f^{(k,m+1)}(x,s) - P_n^{(n-k)}(a) f^{(k,m+1)}(a,s)\bigr]\,\mathrm{d}s \\ &+ \sum_{i=1}^{m} \frac{(-1)^{n+i+1}}{m!n!a_nb_m} \int_a^x P_n(t) \bigl[Q_m^{(m-i)}(y) f^{(n+1,i)}(t,y) - Q_m^{(m-i)}(b) f^{(n+1,i)}(t,b)\bigr]\,\mathrm{d}t \\ &+ \frac{(-1)^{m+n}}{m!n!a_nb_m} \int_a^x \int_b^y P_n(t) Q_m(s) f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s. \end{aligned} \tag{2.6}$$

The proof is complete.

3. Some special cases of Theorem 1

In this section, we will deduce some special cases of Theorem 1.

Definition 1 ([2])**.** Let $P_k(t)$ be a polynomial such that

$$P_k'(t) = P_{k-1}(t), \quad P_0(t) = 1, \quad k = 1, 2, \dots, \tag{3.1}$$

then we call $P_k(t)$ a harmonic polynomial or an Appell polynomial.

It easy to see that the following proposition holds.

Proposition 1. *Let $P_n(t)$ be an Appell polynomial of degree n and the coefficient of the term t^n equal a_n. Then $a_n = \frac{1}{n!}$.*

Theorem 2 (Harmonic Darboux's formula of functions with two variables)**.** *Let $D\subset\mathbb{R}^2$ be a convex region and $(a,b)\in D$. Define $f:D\to\mathbb{R}$ such that $f^{(i,j)}(x,y)$ is continuous on D for $0\le i\le n$ and $0\le j\le m$. Further, let $P_n(t)$ and $Q_m(s)$ be two harmonic polynomials. Then*

$$f(x,y)=f(a,b)+C(f,P_n,Q_m)+D(f,P_n,Q_m)+S(f,P_n,Q_m)+T(f,P_n,Q_m), \tag{3.2}$$

where

$$\begin{aligned}C(f,P_n,Q_m)={}&\sum_{k=1}^{n}(-1)^{k+1}\Big[P_k(x)f^{(k,0)}(x,b)-P_k(a)f^{(k,0)}(a,b)\Big]\\&+\sum_{i=1}^{m}(-1)^{i+1}\Big[Q_i(y)f^{(0,i)}(a,y)-Q_i(b)f^{(0,i)}(a,b)\Big],\end{aligned} \tag{3.3}$$

$$\begin{aligned}D(f,P_n,Q_m)={}&\sum_{k=1}^{n}\sum_{i=1}^{m}(-1)^{k+i}P_k(x)\Big[Q_i(y)f^{(k,i)}(x,y)-Q_i(b)f^{(k,i)}(x,b)\Big]\\&-\sum_{k=1}^{n}\sum_{i=1}^{m}(-1)^{k+i}P_k(a)\Big[Q_i(y)f^{(k,i)}(a,y)-Q_i(b)f^{(k,i)}(a,b)\Big],\end{aligned} \tag{3.4}$$

$$\begin{aligned}S(f,P_n,Q_m)={}&(-1)^n\int_a^x P_n(t)f^{(n+1,0)}(t,b)\,\mathrm{d}t+(-1)^m\int_b^y Q_m(s)f^{(0,m+1)}(a,s)\,\mathrm{d}s\\&+\sum_{k=1}^{n}(-1)^{m+k+1}\int_b^y Q_m(s)\Big[P_k(x)f^{(k,m+1)}(x,s)-P_k(a)f^{(k,m+1)}(a,s)\Big]\,\mathrm{d}s\\&+\sum_{i=1}^{m}(-1)^{n+i+1}\int_a^x P_n(t)\Big[Q_i(y)f^{(n+1,i)}(t,y)-Q_i(b)f^{(n+1,i)}(t,b)\Big]\,\mathrm{d}t,\end{aligned} \tag{3.5}$$

$$T(f,P_n,Q_m)=(-1)^{m+n}\int_a^x\int_b^y P_n(t)Q_m(s)f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s. \tag{3.6}$$

Proof. Let $P_n(t)$ and $Q_m(s)$ be harmonic polynomials and

$$a_n=\frac{1}{n!},\quad b_m=\frac{1}{m!},\quad P_n^{(n-j)}(t)=P_j(t),\quad Q_i(s)=Q_i(s) \tag{3.7}$$

in Theorem 1, then Theorem 2 follows. □

Theorem 3. *Let $D\subset\mathbb{R}^2$ be a convex region, $(a,b)\in D$, and $f:D\to\mathbb{R}$ satisfy that $f^{(i,j)}(x,y)$ is continuous on D for $0\le i\le n$ and $0\le j\le m$. For $0\le\lambda\le1$ and $0\le\mu\le1$, we have*

$$f(x,y)=f(a,b)+C(f,\lambda,\mu)+D(f,\lambda,\mu)+S(f,\lambda,\mu)+T(f,\lambda,\mu), \tag{3.8}$$

where

$$\begin{aligned}C(f,\lambda,\mu)={}&\sum_{k=1}^{n}\frac{(a-x)^k}{k!}\big[(\lambda-1)^kf^{(k,0)}(a,b)-\lambda^kf^{(k,0)}(x,b)\big]\\&+\sum_{i=1}^{m}\frac{(b-y)^i}{i!}\big[(\mu-1)^if^{(0,i)}(a,b)-\mu^if^{(0,i)}(a,y)\big],\end{aligned} \tag{3.9}$$

$$\begin{aligned}D(f,\lambda,\mu)={}&\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{(a-x)^k(b-y)^i\lambda^k}{k!i!}\big[\mu^if^{(k,i)}(x,y)-(\mu-1)^if^{(k,i)}(x,b)\big]\\&-\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{(a-x)^k(b-y)^i(\lambda-1)^k}{k!i!}\big[\mu^if^{(k,i)}(a,y)-(\mu-1)^if^{(k,i)}(a,b)\big],\end{aligned} \tag{3.10}$$

$$\begin{aligned}S(f,\lambda,\mu)={}&\frac{(-1)^n}{n!}\int_a^x[t-(\lambda a+(1-\lambda)x)]^nf^{(n+1,0)}(t,b)\,\mathrm{d}t\\&+\frac{(-1)^m}{m!}\int_b^y[s-(\mu b+(1-\mu)y)]^mf^{(0,m+1)}(a,s)\,\mathrm{d}s\\&+\sum_{k=1}^{n}\frac{(a-x)^k}{m!k!}\int_b^y[(\mu b+(1-\mu)y)-s]^m\big[(\lambda-1)^kf^{(k,m+1)}(a,s)-\lambda^kf^{(k,m+1)}(x,s)\big]\,\mathrm{d}s\\&+\sum_{i=1}^{m}\frac{(b-y)^i}{n!i!}\int_a^x[(\lambda a+(1-\lambda)x)-t]^n\big[(\mu-1)^if^{(n+1,i)}(t,b)-\mu^if^{(n+1,i)}(t,y)\big]\,\mathrm{d}t,\end{aligned} \tag{3.11}$$

$$T(f,\lambda,\mu)=\frac{1}{m!n!}\int_a^x\int_b^y[(\lambda a+(1-\lambda)x)-t]^n[(\mu b+(1-\mu)y)-s]^mf^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s. \tag{3.12}$$

Proof. Taking $P_n(t) = [t - (\lambda a + (1-\lambda)x)]^n$ and $Q_m(s) = [s - (\mu b + (1-\mu)y)]^m$ and noticing that

$$\begin{aligned} &a_n = 1, && b_m = 1, \\ &P_n^{(n-k)}(t) = \frac{n!}{k!}[t - (\lambda a + (1-\lambda)x)]^k, && Q_m^{(m-i)}(s) = \frac{m!}{i!}[s - (\mu b + (1-\mu)y)]^i \end{aligned} \tag{3.13}$$

in Theorem 1 leads to Theorem 3 straightforwardly. □

Remark 3. If letting $P_n(t) = (t-x)^n$ and $Q_m = (s-y)^m$ in Theorem 1, or equivalently, taking $\lambda = 0$ and $\mu = 0$ in Theorem 3, then A. Sard's formula of functions with two independent variables, Theorem C, follows. Therefore, Theorem 1 generalizes the noted A. Sard's formula for functions with two independent variables, Theorem C, which is also stated in [3] and [11, p. 138 and p. 157].

Definition 2 ([1, 23.1.1])**.** Bernoulli's polynomials $B_k(x)$ for k being nonnegative integers are defined by

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x), \quad |x| < 2\pi, \quad t \in \mathbb{R}, \tag{3.14}$$

where $B_k(0) = B_k$ is called Bernoulli's numbers.

Definition 3 ([1, 23.1.1])**.** Euler's polynomials $E_k(x)$ for k being nonnegative integers are defined by

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_k(x), \quad |x| < \pi, \quad t \in \mathbb{R}, \tag{3.15}$$

where $2^k E_k(\frac{1}{2}) = E_k$ is called Euler's numbers.

Remark 4. Notice that Bernoulli's numbers and polynomials and Euler's numbers and polynomials have been generalized by the authors in [4, 5, 6, 7, 9] recently.

Lemma 1 ([1, 23.1.5] and [8])**.** *The following identities hold*

$$B_k'(x) = kB_{k-1}(x), \quad E_k'(x) = kE_{k-1}(x), \quad k = 1, 2, \dots. \tag{3.16}$$

Lemma 2 ([1, 23.1.6] and [8])**.** *The following identities hold*

$$B_i(t+1) - B_i(t) = it^{i-1}, \quad E_i(t+1) + E_i(t) = 2t^i, \quad i = 0, 1, \dots. \tag{3.17}$$

Lemma 3 ([1, 23.1.20] and [8])**.** *The following identities hold*

$$B_k(0) = (-1)^k B_k(1) = B_k, \quad k = 0, 1, 2, \dots, \tag{3.18}$$

$$E_i(0) = -E_i(1) = -\frac{2}{i+1}(2^{i+1} - 1)B_{i+1}, \quad i = 1, 2, \dots. \tag{3.19}$$

Theorem 4. *Let $D \subset \mathbb{R}^2$ be a convex plane region and $(a,b) \in D$. Define $f : D \to \mathbb{R}$ such that $f^{(i,j)}(x,y)$ is continuous on D for $0 \le i \le n$ and $0 \le j \le m$. Then*

$$f(x,y) = f(a,b) + C(f, B_n, E_m) + D(f, B_n, E_m) + S(f, B_n, E_m) + T(f, B_n, E_m), \tag{3.20}$$

where

$$\begin{aligned} C(f, B_n, E_m) = &\sum_{k=1}^{n} \frac{(a-x)^k}{k!} B_k \left[f^{(k,0)}(a,b) + (-1)^{k+1} f^{(k,0)}(x,b)\right] \\ &- \sum_{i=1}^{m} \frac{2(b-y)^i (2^{i+1} - 1)}{(i+1)!} B_{i+1} \left[f^{(0,i)}(a,b) + f^{(0,i)}(a,y)\right], \end{aligned} \tag{3.21}$$

$$\begin{aligned} D(f, B_n, E_m) = &\sum_{k=1}^{n} \sum_{i=1}^{m} \frac{2(a-x)^k (b-y)^i (2^{i+1} - 1)}{k!(i+1)!} \\ &\times B_k B_{i+1} \left[(-1)^k f^{(k,i)}(x,y) + (-1)^k f^{(k,i)}(x,b) - f^{(k,i)}(a,y) - f^{(k,i)}(a,b)\right], \end{aligned} \tag{3.22}$$

$$\begin{aligned} S(f, B_n, E_m) = &\frac{(a-x)^n}{n!} \int_a^x B_n\left(\frac{t-a}{x-a}\right) f^{(n+1,0)}(t,b) \,\mathrm{d}t \\ &+ \frac{(b-y)^m}{m!} \int_b^y E_m\left(\frac{s-b}{y-b}\right) f^{(0,m+1)}(a,s) \,\mathrm{d}s \\ &+ \sum_{k=1}^{n} \frac{(a-x)^k (b-y)^m}{m!k!} B_k \int_b^y E_m\left(\frac{s-b}{y-b}\right) \left[f^{(k,m+1)}(a,s) + (-1)^{k+1} f^{(k,m+1)}(x,s)\right] \mathrm{d}s \\ &- \sum_{i=1}^{m} \frac{2(a-x)^n (b-y)^i (2^{i+1} - 1)}{n!(i+1)!} B_{i+1} \int_a^x B_n\left(\frac{t-a}{x-a}\right) \left[f^{(n+1,i)}(t,b) + f^{(n+1,i)}(t,y)\right] \mathrm{d}t, \end{aligned} \tag{3.23}$$

$$T(f,B_n,E_m)=\frac{(a-x)^n(b-y)^m}{m!n!}\int_a^x\int_b^y B_n\Big(\frac{t-a}{x-a}\Big)E_m\Big(\frac{s-b}{y-b}\Big)f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s. \tag{3.24}$$

Proof. Taking

$$P_n(t)=\frac{(x-a)^n}{n!}B_n\Big(\frac{t-a}{x-a}\Big),\quad Q_m(s)=\frac{(y-b)^m}{m!}E_m\Big(\frac{s-b}{y-b}\Big) \tag{3.25}$$

in Theorem 1 and considering Lemma 1, Lemma 2, Lemma 3 and

$$a_n=\frac{1}{n!}, \qquad b_m=\frac{1}{m!},$$
$$P_n^{(n-k)}(t)=\frac{(x-a)^k}{k!}B_k\Big(\frac{t-a}{x-a}\Big), \qquad Q_m^{(m-i)}(s)=\frac{(y-b)^i}{i!}E_i\Big(\frac{s-b}{y-b}\Big).$$

yields Theorem 4 easily. □

Remark 5. It is easy to verify that the polynomials

$$P_n(t)=\frac{(x-a)^n}{n!}B_n\Big(\frac{t-a}{x-a}\Big),\quad Q_m(s)=\frac{(y-b)^m}{m!}E_m\Big(\frac{s-b}{y-b}\Big) \tag{3.26}$$

satisfy Appell conditions

$$P_n'(t)=P_{n-1}(t),\qquad Q_m'(s)=Q_{m-1}(s). \tag{3.27}$$

Thus, the polynomials $P_n(t)$ and $Q_m(s)$ in Theorem 4 are Appell's polynomials.

If in Theorem 1 taking

$$P_n(t)=\frac{(x-a)^n}{n!}B_n\Big(\frac{t-a}{x-a}\Big),\quad Q_m(s)=\frac{(y-b)^m}{m!}B_m\Big(\frac{s-b}{y-b}\Big) \tag{3.28}$$

or

$$P_n(t)=\frac{(x-a)^n}{n!}E_n\Big(\frac{t-a}{x-a}\Big),\quad Q_m(s)=\frac{(y-b)^m}{m!}E_m\Big(\frac{s-b}{y-b}\Big), \tag{3.29}$$

which are harmonic polynomials, then we can obtain more other important special cases of Theorem 1.

Theorem 5 (Obreschkoff's formula of functions with two variables)**.** *Let $D\subset\mathbb{R}^2$ be a convex region, $(a,b)\in D$, and $f:D\to\mathbb{R}$ be a function such that $f^{(m+n,p+q)}(x,y)$ is absolutely continuous on D. Then*

$$\begin{aligned}
f(x,y)={}&\sum_{k=1}^{n}\sum_{i=1}^{q}\frac{\binom{n}{k}\binom{q}{i}}{\binom{m+n}{k}\binom{p+q}{i}}(a-x)^k(b-y)^i f^{(k,i)}(x,y)\\
&-\sum_{k=1}^{n}\sum_{i=0}^{p}\frac{\binom{n}{k}\binom{p}{i}}{\binom{m+n}{k}\binom{p+q}{i}}(a-x)^k(y-b)^i f^{(k,i)}(x,b)\\
&-\sum_{k=0}^{m}\sum_{i=1}^{q}\frac{\binom{m}{k}\binom{q}{i}}{\binom{m+n}{k}\binom{p+q}{i}}(x-a)^k(b-y)^i f^{(k,i)}(a,y)\\
&+\sum_{k=0}^{m}\sum_{i=0}^{p}\frac{\binom{m}{k}\binom{p}{i}}{\binom{m+n}{k}\binom{p+q}{i}}(x-a)^k(y-b)^i f^{(k,i)}(a,b)\\
&+\sum_{k=0}^{m}\frac{\binom{m}{k}}{(p+q)!\binom{m+n}{k}}(x-a)^k\int_b^y(y-s)^p(b-s)^q f^{(k,p+q+1)}(a,s)\,\mathrm{d}s \qquad (3.30)\\
&-\sum_{k=1}^{n}\frac{\binom{n}{k}}{(p+q)!\binom{m+n}{k}}(a-x)^k\int_b^y(y-s)^p(b-s)^q f^{(k,p+q+1)}(x,s)\,\mathrm{d}s\\
&+\sum_{i=0}^{p}\frac{\binom{p}{i}}{(m+n)!\binom{p+q}{i}}(y-b)^i\int_a^x(x-t)^m(a-t)^n f^{(m+n+1,i)}(t,b)\,\mathrm{d}t\\
&-\sum_{i=1}^{q}\frac{\binom{q}{i}}{(m+n)!\binom{p+q}{i}}(b-y)^i\int_a^x(x-t)^m(a-t)^n f^{(m+n+1,i)}(t,y)\,\mathrm{d}t\\
&+\frac{1}{(m+n)!(p+q)!}\int_a^x\int_b^y(x-t)^m(a-t)^n(y-s)^p(b-s)^q f^{(m+n+1,p+q+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s, \qquad (3.31)
\end{aligned}$$

where m,n,p,q are nonnegative integers.

Proof. Taking $P_{m+n}(t) = (t-x)^m(t-a)^n$ and $Q_{p+q}(s) = (s-y)^p(s-b)^q$ and replacing n by $m+n$, m by $p+q$ in Theorem 1, considering $a_{m+n} = 1$ and $b_{p+q} = 1$, and emploiting Leibnitz's formula for derivative yields

$$P_{m+n}^{(m+n-k)}(x) = \begin{cases} \binom{m+n-k}{m}\dfrac{m!n!}{k!}(x-a)^k, & k = 0, 1, \cdots, n, \\ 0, & k = n+1, \cdots, n+m, \end{cases} \tag{3.32}$$

$$P_{m+n}^{(m+n-k)}(a) = \begin{cases} \binom{m+n-k}{n}\dfrac{m!n!}{k!}(a-x)^k, & k = 0, 1, \cdots, m, \\ 0, & k = m+1, \cdots, m+n, \end{cases} \tag{3.33}$$

$$Q_{p+q}^{(p+q-i)}(y) = \begin{cases} \binom{p+q-i}{p}\dfrac{p!q!}{i!}(y-b)^i, & i = 0, 1, \cdots, q, \\ 0, & i = q+1, \cdots, q+p, \end{cases} \tag{3.34}$$

$$Q_{p+q}^{(p+q-i)}(b) = \begin{cases} \binom{p+q-i}{q}\dfrac{p!q!}{i!}(b-y)^i, & i = 0, 1, \cdots, p, \\ 0, & i = p+1, \cdots, p+q, \end{cases} \tag{3.35}$$

further, combining the following combinational identities

$$\binom{m+n-k}{m}\frac{m!n!}{(m+n)!} = \frac{\binom{n}{k}}{\binom{m+n}{k}}, \qquad \binom{m+n-k}{n}\frac{m!n!}{(m+n)!} = \frac{\binom{m}{k}}{\binom{m+n}{k}}, \tag{3.36}$$

$$\binom{p+q-i}{p}\frac{p!q!}{(p+q)!} = \frac{\binom{q}{i}}{\binom{p+q}{i}}, \qquad \binom{p+q-i}{q}\frac{p!q!}{(p+q)!} = \frac{\binom{p}{i}}{\binom{p+q}{i}}, \tag{3.37}$$

with Theorem 1, then Theorem 4 follows. □

4. Estimates of remainders

In this section, we shall give some estimates for remainders stated in above sections. The following lemmas are necessary.

Lemma 4 ([1, 23.1.12]). *We have the following*

$$\int_0^1 B_n(x)B_m(x)\,\mathrm{d}x = (-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n}, \quad m, n = 1, 2, \dots, \tag{4.1}$$

$$\int_0^1 E_n(x)E_m(x)\,\mathrm{d}x = 4(-1)^n(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2}, \quad m, n = 0, 1, \dots. \tag{4.2}$$

Theorem 6. *Under conditions of Theorem 1, the remainder (1.10) can be estimated as follows:*

$$|T(f, P_n, Q_m)| \le \frac{1}{m!n!\,|a_nb_m|}\max_{t\in[a,x]}\{|P_n(t)|\}\max_{y\in[b,y]}\{|Q_m(s)|\}\left|\int_a^x\int_b^y f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s\right|, \tag{4.3}$$

$$|T(f, P_n, Q_m)| \le \frac{1}{m!n!\,|a_nb_m|}\max_{(t,s)\in[a,x]\times[b,y]}\left\{\left|f^{(n+1,m+1)}(t,s)\right|\right\}\left|\int_a^x P_n(t)\,\mathrm{d}t\right|\left|\int_b^y Q_m(s)\,\mathrm{d}s\right|, \tag{4.4}$$

$$|T(f, P_n, Q_m)| \le \frac{(x-a)(y-b)}{m!n!\,|a_nb_m|}\max_{t\in[a,x]}\{|P_n(t)|\}\max_{s\in[b,y]}\{|Q_m(s)|\}\max_{(t,s)\in[a,x]\times[b,y]}\left\{\left|f^{(n+1,m+1)}(t,s)\right|\right\}, \tag{4.5}$$

$$|T(f, P_n, Q_m)| \le \frac{1}{m!n!\,|a_nb_m|}\left[\int_a^x|P_n(t)|^q\,\mathrm{d}t\int_b^y|Q_m(s)|^q\,\mathrm{d}s\right]^{\frac{1}{q}}\left[\int_a^x\int_b^y\left|f^{(n+1,m+1)}(t,s)\right|^p\mathrm{d}t\,\mathrm{d}s\right]^{\frac{1}{p}}. \tag{4.6}$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The estimates (4.3), (4.4) and (4.5) is straightforward. The estimate (4.6) can be obtained by Hölder's inequality of double integral. □

Theorem 7. *Under conditions of Theorem 3, we have the following estimates for the remainder (3.12):*

$$|T(f,\lambda,\mu)| \le \frac{(x-a)^n(y-b)^m}{m!n!}\left|\int_a^x\int_b^y f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s\right|, \tag{4.7}$$

$$\begin{aligned}|T(f,\lambda,\mu)| \le{}& \frac{(x-a)^{n+1}(y-b)^{m+1}\left|\lambda^{n+1}-(\lambda-1)^{n+1}\right|\left|\mu^{m+1}-(\mu-1)^{m+1}\right|}{(m+1)!(n+1)!}\\ &\times\max_{(t,s)\in[a,x]\times[b,y]}\left\{\left|f^{(n+1,m+1)}(t,s)\right|\right\},\end{aligned} \tag{4.8}$$

$$
\begin{aligned}
|T(f,\lambda,\mu)| \le{}& \frac{(x-a)^{n+\frac{1}{q}}(y-b)^{m+\frac{1}{q}}[\lambda^{nq+1}+(1-\lambda)^{nq+1}]^{\frac{1}{q}}[\mu^{mq+1}+(1-\mu)^{mq+1}]^{\frac{1}{q}}}{m!n![(nq+1)(mq+1)]^{\frac{1}{q}}} \\
&\times\left[\int_a^x\int_b^y \left|f^{(n+1,m+1)}(t,s)\right|^p \,\mathrm{d}t\,\mathrm{d}s\right]^{\frac{1}{p}},
\end{aligned}
\tag{4.9}
$$

where $p>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Taking $P_n(t)=[t-(\lambda a+(1-\lambda)x)]^n$ and $Q_m(s)=[s-(\mu b+(1-\mu)y)]^m$ in (4.3) and (4.4) and considering $a_n=1$ and $b_m=1$ yields estimates (4.7) and (4.8). The estimate (4.9) follows from Hölder's inequality of double integral. □

Theorem 8. *Under conditions of Theorem 4, the remainder* (3.24) *can be estimated as follows:*

$$
\begin{aligned}
|T(f,B_n,E_m)| \le{}& \frac{(x-a)^n(y-b)^m}{m!n!}\max_{t\in[a,x]}\left\{\left|B_n\left(\frac{t-a}{x-a}\right)\right|\right\}\max_{s\in[b,y]}\left\{\left|E_m\left(\frac{s-b}{y-b}\right)\right|\right\} \\
&\times\left|\int_a^x\int_b^y f^{(n+1,m+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s\right|,
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
|T(f,B_n,E_m)| \le{}& \frac{(x-a)^{n+1}(y-b)^{m+1}}{2}\left[\frac{n!}{m!(2n)!}|B_{2n}|+\frac{4m!(4^{m+1}-1)}{n!(2m+2)!}|B_{2m+2}|\right] \\
&\times\max_{(t,s)\in[a,x]\times[b,y]}\left\{\left|f^{(n+1,m+1)}(t,s)\right|\right\}.
\end{aligned}
\tag{4.11}
$$

Proof. The inequality (4.10) can be obtained directly.

Taking $m=n$ in Lemma 4 gives us

$$
\begin{aligned}
&\int_0^1 B_n^2(x)\,\mathrm{d}x=(-1)^{n-1}\frac{(n!)^2}{(2n)!}B_{2n}=\frac{(n!)^2}{(2n)!}|B_{2n}|, \\
&\int_0^1 E_m^2(x)\,\mathrm{d}x=\frac{4(-1)^m(4^{m+1}-1)(m!)^2}{(2m+2)!}B_{2m+2}=\frac{4(4^{m+1}-1)(m!)^2}{(2m+2)!}|B_{2m+2}|.
\end{aligned}
\tag{4.12}
$$

Considering (4.12) yields

$$
\begin{aligned}
|T(f,B_n,E_m)| &\le \frac{(x-a)^n(y-b)^m}{m!n!}\int_a^x\int_b^y\left|B_n\left(\frac{t-a}{x-a}\right)\right|\left|E_m\left(\frac{s-b}{y-b}\right)\right|\left|f^{(n+1,m+1)}(t,s)\right|\mathrm{d}t\,\mathrm{d}s \\
&\le \frac{(x-a)^n(y-b)^m}{2m!n!}\int_a^x\int_b^y\left[B_n^2\left(\frac{t-a}{x-a}\right)+E_m^2\left(\frac{s-b}{y-b}\right)\right]\left|f^{(n+1,m+1)}(t,s)\right|\mathrm{d}t\,\mathrm{d}s \\
&\le \frac{(x-a)^{n+1}(y-b)^{m+1}}{2m!n!}\left[\int_0^1 B_n^2(x)\,\mathrm{d}x+\int_0^1 E_m^2(y)\,\mathrm{d}y\right]\max_{(t,s)\in[a,x]\times[b,y]}\left\{\left|f^{(n+1,m+1)}(t,s)\right|\right\} \\
&\le \frac{(x-a)^{n+1}(y-b)^{m+1}}{2}\left[\frac{n!}{m!(2n)!}|B_{2n}|+\frac{4m!(4^{m+1}-1)}{n!(2m+2)!}|B_{2m+2}|\right]\max_{(t,s)\in[a,x]\times[b,y]}\left\{\left|f^{(n+1,m+1)}(t,s)\right|\right\}.
\end{aligned}
$$

The proof is complete. □

Theorem 9. *Under conditions of Theorem 5, the remainder* (3.31) *can be estimated as follows:*

$$
\begin{aligned}
&\left|\frac{1}{(m+n)!(p+q)!}\int_a^x\int_b^y (x-t)^m(a-t)^n(y-s)^p(b-s)^q f^{(m+n+1,p+q+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s\right| \\
&\quad\le \frac{m^mn^np^pq^q}{(m+n)!(p+q)!}\left(\frac{x-a}{m+n}\right)^{m+n}\left(\frac{y-b}{p+q}\right)^{p+q}\left|\int_a^x\int_b^y f^{(m+n+1,p+q+1)}(t,s)\,\mathrm{d}t\,\mathrm{d}s\right|.
\end{aligned}
\tag{4.13}
$$

Proof. This follows from (4.3) easily. □

5. Examples

Let $f(x,y)=\ln(x+y)$. By direct compution, we have

$$
f^{(n,m)}(x,y)=\frac{(-1)^{m+n-1}(m+n-1)!}{(x+y)^{m+n}}.
\tag{5.1}
$$

Example 1. In Theorem 3, let $D=(0,+\infty)\times(0,+\infty)$, $(a,b)\in D$ and $f(x,y)=\ln(x+y)$. Considering (5.1) gives

$$
\ln(x+y)=\ln(a+b)+\sum_{k=1}^n\frac{(x-a)^k}{k}\left[\left(\frac{\lambda}{x+b}\right)^k-\left(\frac{\lambda-1}{a+b}\right)^k\right]+\sum_{i=1}^m\frac{(y-b)^i}{i}\left[\left(\frac{\mu}{a+y}\right)^i-\left(\frac{\mu-1}{a+b}\right)^i\right]
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{\lambda^k(k+i-1)!(x-a)^k(y-b)^i}{k!i!}\left[\frac{(\mu-1)^i}{(x+b)^{k+i}}-\frac{\mu^i}{(x+y)^{k+i}}\right]\\
&-\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{(\lambda-1)^k(k+i-1)!(x-a)^k(y-b)^i}{k!i!}\left[\frac{(\mu-1)^i}{(a+b)^{k+i}}-\frac{\mu^i}{(a+y)^{k+i}}\right]\\
&+\int_a^x\frac{[t-(\lambda a+(1-\lambda)x)]^n}{(t+b)^{n+1}}\,\mathrm{d}t+\int_b^y\frac{[s-(\mu b+(1-\mu)y)]^m}{(a+s)^{m+1}}\,\mathrm{d}s \qquad (5.2)\\
&+\sum_{k=1}^{n}\frac{(m+k)!(x-a)^k}{m!k!}\int_b^y[s-(\mu b+(1-\mu)y)]^m\left[\frac{(\lambda-1)^k}{(a+s)^{m+k+1}}-\frac{\lambda^k}{(x+s)^{m+k+1}}\right]\mathrm{d}s\\
&+\sum_{i=1}^{m}\frac{(n+i)!(y-b)^i}{n!i!}\int_a^x[t-(\lambda a+(1-\lambda)x)]^n\left[\frac{(\mu-1)^i}{(t+b)^{n+i+1}}-\frac{\mu^i}{(t+y)^{n+i+1}}\right]\mathrm{d}t\\
&-\frac{(m+n+1)!}{m!n!}\int_a^x\int_b^y\frac{[t-(\lambda a+(1-\lambda)x)]^n[s-(\mu b+(1-\mu)y)]^m}{(t+s)^{m+n+2}}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

Example 2. In Example 1, letting $\lambda=0$ and $\mu=0$, then we obtain the following A. Sard's expasion of the function $\ln(x+y)$:

$$
\begin{aligned}
\ln(x+y)=\ln(a+b)&-\sum_{k=0}^{n}\sum_{i=0}^{m}\frac{(k+i-1)!(a-x)^k(b-y)^i}{k!i!(a+b)^{k+i}}+\sum_{k=0}^{n}\frac{(m+k)!(a-x)^k}{m!k!}\int_b^y\frac{(s-y)^m}{(a+s)^{m+k+1}}\,\mathrm{d}s\\
&+\sum_{i=0}^{m}\frac{(n+i)!(b-y)^i}{n!i!}\int_a^x\frac{(t-x)^n}{(t+b)^{n+i+1}}\,\mathrm{d}t-\frac{(m+n+1)!}{m!n!}\int_a^x\int_b^y\frac{(t-x)^n(s-y)^m}{(t+s)^{m+n+2}}\,\mathrm{d}t\,\mathrm{d}s. \qquad (5.3)
\end{aligned}
$$

Example 3. In Theorem 4, letting $D=(0,+\infty)\times(0,+\infty)$, $(a,b)\in D$ and $f(x,y)=\ln(x+y)$ and considering (5.1) yields

$$
\begin{aligned}
&\ln(x+y)=\ln(a+b)\\
&+\sum_{k=1}^{n}\frac{(x-a)^kB_k}{k}\left[\left(\frac{-1}{x+b}\right)^k-\frac{1}{(a+b)^k}\right]+\sum_{i=1}^{m}\frac{2(2^{i+1}-1)(y-b)^iB_{i+1}}{i(i+1)}\left[\frac{1}{(a+y)^i}+\frac{1}{(a+b)^i}\right]\\
&+\sum_{k=1}^{n}\sum_{i=1}^{m}\frac{2(k+i-1)!(2^{i+1}-1)(x-a)^k(y-b)^iB_{i+1}B_k}{k!(i+1)!}\left[\frac{(-1)^{k+1}}{(x+y)^{k+i}}+\frac{(-1)^{k+1}}{(x+b)^{k+i}}\right.\\
&\left.+\frac{1}{(a+y)^{k+i}}+\frac{1}{(a+b)^{k+i}}\right]+\int_a^x\frac{(x-a)^n}{(t+b)^{n+1}}B_n\left(\frac{t-a}{x-a}\right)\mathrm{d}t+\int_b^y\frac{(y-b)^m}{(a+s)^{m+1}}E_m\left(\frac{s-b}{y-b}\right)\mathrm{d}s \qquad (5.4)\\
&+\sum_{k=1}^{n}\frac{(m+k)!(x-a)^k(y-b)^mB_k}{m!k!}\int_b^yE_m\left(\frac{s-b}{y-b}\right)\left[\frac{1}{(a+s)^{m+k+1}}-\frac{(-1)^k}{(x+s)^{m+k+1}}\right]\mathrm{d}s\\
&-\sum_{i=1}^{m}\frac{2(n+i)!(2^{i+1}-1)(x-a)^n(y-b)^iB_{i+1}}{n!(i+1)!}\int_a^xB_n\left(\frac{t-a}{x-a}\right)\left[\frac{1}{(t+b)^{n+i+1}}+\frac{1}{(t+y)^{n+i+1}}\right]\mathrm{d}t\\
&-\frac{(m+n+1)!(x-a)^n(y-b)^m}{m!n!}\int_a^x\int_b^y\frac{B_n(\frac{t-a}{x-a})E_m(\frac{s-b}{y-b})}{(t+s)^{m+n+2}}\,\mathrm{d}t\,\mathrm{d}s.
\end{aligned}
$$

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